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ELLPACK '78 User's Guide-Preliminary Version

John R. Rice
Purdue University, jrr@cs.purdue.edu

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ELLPACK 78 USER'S GUIDE - PRELIMINARY VERSION

John R. Rice

Purdue University

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ABSTRACT

This report outlines the additional features in ELLPACK 78 compared to ELLPACK 77. It is assumed that the reader is familiar with ELLPACK 77 (CSD-TR 289). ELLPACK 78 provides facilities for general 2-dimensional domains and the internal representations of the geometry information is described in Appendix A.

CONTENTS

ELLPACK 78 USER'S GUIDE

John R. Rice

1. INTRODUCTION.

The ELLPACK 78 version of ELLPACK adds the capability to handle general geometry in two dimensions. Both versions are in the same system and ELLPACK 77 programs will continue to operate unchanged. The principal changes for ELLPACK 78 are:

(a) Definition and processing of the geometry information

(b) Discretization of the operator in general domains.

The first changes affect the ELLPACK system and they have been carried out. The second affect contributed modules for the DISCRETIZATION segment. Two such modules are already installed (5 POINT STAR and 2DEPEP) and two more (HODIE ACF and P3C1 COLLOCATION) are in progress. The INDEXING and SOLUTION modules should, by and large, be unaffected by the geometry. There are additions to the OUTPUT segment as well as changes in the operation of some OUTPUT statements. Note that the same ELLPACK statements can invoke different routines in ELLPACK 77 or ELLPACK 78, e.g. 5-POINT STAR and PLOT are implemented by entirely different programs in the two cases.

Figure 1 shows a simple ELLPACK 77 program and its equivalent in ELLPACK 78. The principal change is that the BOUNDARY segment is split into two segments (DOMAIN and BOUNDARY). In ELLPACK 78, the DOMAIN specifies the geometry and BOUNDARY specifies the boundary conditions. The connection is made between the domain definition and the boundary conditions by numbering the boundary pieces and giving conditions according to this numbering.

```
*
*        EXAMPLE PROGRAM FOR ELLPACK 78 USERS GUIDE
EQUATION.    2 DIMENSIONS  $  CONSTANT COEFFICIENTS
         UXX$ + UYY$ + 3.UX$ - 4. U  = EXP(X+Y)*SIN(X)
*
BOUNDARY.  X = 0.0 ,  U = 0.0   $  Y = 2.0 ,  U = X
           X = 1.0 ,  U = Y/2. $ Y = -1.0 , U = SIN(3.1415926536*X) - X/2.
*
*
GRID.       UNIFORM X = 4  $  UNIFORM Y = 10
*
DISCRETIZATION.    5-POINT STAR
INDEXING.          NATURAL
SOLUTION.          LINPACK BAND
OUTPUT.            TABLE-SOLUTION  $  PLOT-SOLUTION
OPTIONS.           TIME $ MEMORY
END.
```

```
*
*        EXAMPLE PROGRAM FOR ELLPACK 78 USERS GUIDE
*        CHANGED TO ELLPACK 78 FORM
EQUATION.    2 DIMENSIONS  $  CONSTANT COEFFICIENTS
         UXX$ + UYY$ + 3.UX$ - 4. U  = EXP(X+Y)*SIN(X)
*
DOMAIN.
        1.  X = 0.0           $  Y = T             $ T = -1. TO 2.0
        2.  X = R             $  Y = 2.            $  R = 0.0 TO 1.0
        3. TO 4.  LINE 1.,2.  TO  1.,-1.  TO  0.,-1.
BOUNDARY.     1.  U = 0.0  $  2.  U = X  $ 3.  U = Y/2.
              4.  U = SIN(3.1415926536) - X/2.
*
GRID.         UNIFORM X = 4, 0. TO 1.  $  UNIFORM Y = 10,  -1. TO 2.
*
DISCRETIZATION.    5-POINT STAR
INDEXING.          NATURAL
SOLUTION.          LINPACK BAND
OUTPUT.            TABLE-SOLUTION  $  PLOT-SOLUTION
OPTIONS.           TIME $ MEMORY
END.
```

Figure 1. Top - a simple ELLPACK 77 program. Bottom - an equivalent ELLPACK 78 program.

The keyword RECTANGULAR may be used with the BOUNDARY segment to indicate that ELLPACK 77 is being used but it is not necessary; it does help provide protection from errors.

The remainder of this report gives specific details of the four segments affected: DOMAIN, BOUNDARY, HOLE and OUTPUT.

2. THE DOMAIN SEGMENT.

The domain of the PDE is defined by its boundary which is given in piecewise, parametric form. That is, each piece is defined by

$$x=x(t) \ , \ y=y(t) \quad \text{for } t \in [a,b]$$

which appears in ELLPACK as (if this is the 4th piece)

4. X = X(T) $ Y =Y(T) $ T= 4.1 to 6

The functions X(T) and Y(T) must be ordinary Fortran expressions; complicated definitions are made by defining Fortran functions which are included in the Fortran segment. The limits on the parameter T must be constant expressions e.g. EXP(1.) is allowable to express e, ATAN(1.) for $\pi/4$ and must be increasing

The pieces must be given in order and close to define the domain. The domain is in the interior of the boundary curve and the user must specify the orientation of the boundary: clockwise or counter-clockwise. The keywords CLOCKWISE and COUNTER-CLOCKWISE make this specification with CLOCKWISE the default value.

There is a special simple form for straight line pieces which can be specified just by giving their end points. A sequence of straight lines can be specified by giving the sequence of end points. The keyword LINE indicates this special form and the general syntax for K pieces (starting with the Nth one) is

N. TO (N+K). LINE A_1, B_1 TO A_2, B_2 TO --- TO A_{K+1}, B_{K+1}

Thus pieces 4, 5 and 6 could be specified by

4. to 6. LINE 0,1. TO 0., 2. TO 1.,3. TO 4.2368,4.

Four complete examples of domains and their specifications are given in Figures 2 and 3.

********** WARNING - WARNING **********

The domain must be specified in a reasonable way or the ELLPACK 78 system will not be able to identify all the intersections of the boundary with the pieces. We list certain requirements on the specifications of the domain:

ESSENTIAL REQUIREMENTS

1. Fine Grid The gird must be fine enough that the boundary is well behaved within any particular grid square. The boundary does not intersect any side of a grid square more than twice (and most of them only once).

2. <u>Accurate Boundary with Ordinary Size</u>. The boundary must be given accurately and in units of ordinary size. There is a fixed constant EPSGRD in ELLPACK that is a tolerance for the pieces joining or intersecting the grid lines. (EPSGRD = 10^{-8} for a 14 decimal digit machine, 10^{-6} for 10 digits, 10^{-5} for 8 digits and for 7 digits $5*10^{-5}$. If the units of the x,y coordinate system lead to values of more than 1000 or all less than .001, then there is considerable risk that the system will fail to process the boundary properly.

3. <u>Nice Parameters</u>. The parameters of the pieces must be of ordinary size and should not vary erratically. It is easy to parameterize a simple curve so that ELLPACK does not follow it properly; just make a variations 1 in T correspond to 1 inch on the curve at the start and 10^8 inches at the end.

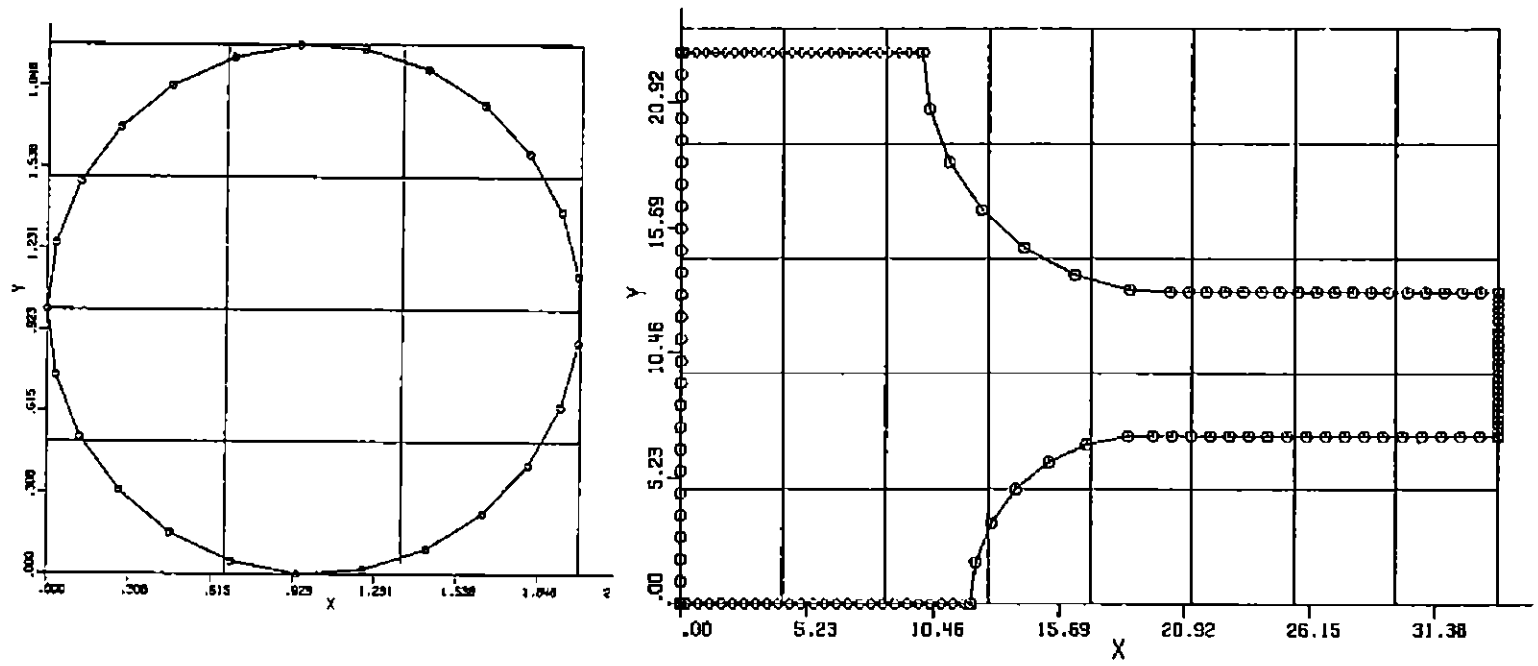
<u>WARNING</u>

4. <u>Protection</u>. The functions defining the pieces should be protected from illegal or nonsense values. A semi-circular piece can be specified by

```
6.  X = T        $Y = SQRT (1.-T**2) $ T = -1. TO 1.
```

This can cause trouble near X= -1 and +1. as initial phases of the numerical method may attempt to use values of X **2 greater than 1. which causes the square root of a negative number to be computed. This will terminate the computation on most systems.

```
*
*               ELLPACK 78 USER GUIDE EXAMPLE 1 - MAY 1979
*
EQUATION.       UXX$ + UYY =  2.*EXP(X+Y)
*                        CIRCULAR DISK
DOMAIN.     1. X = 1.-COS(3.141592654*THETA)
               Y = 1.+SIN(3.141592654*THETA) $ THETA = 0. TO 2.
```

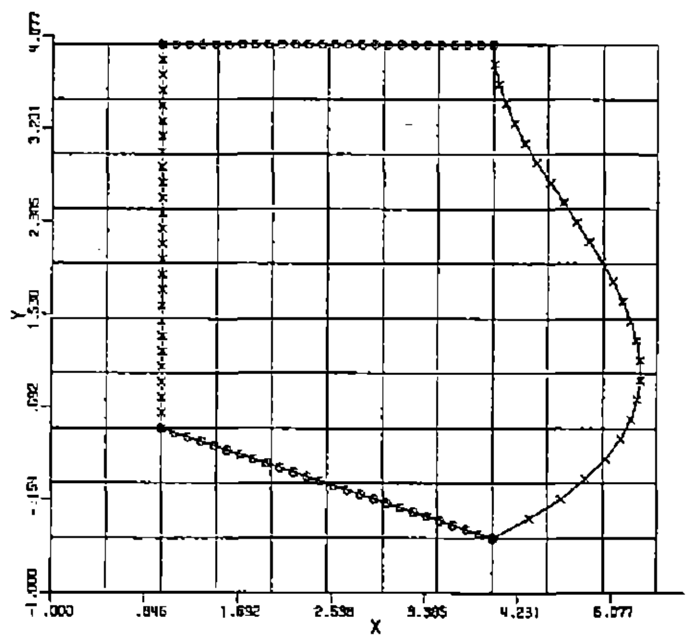



```
*
*          ELLPACK 78 USER GUIDE EXAMPLE 2 - MAY 1979
*       REGION FROM HALL,LUCZAK,SERDY - TOMS 2(1976) 257-274
*
EQUATION.    LAPLACE  $ UXX$ + UYY = 0   $  SELF-ADJOINT
*
DOMAIN. 1. TO 2.  LINE  0,0 TO  0,23 TO 10.,23.
     3. X =XDOM3(P)  $   Y = YDOM3(P)    $ P = 0 TO 19.
     4. LINE  34,13 TO  34,7
     5. X = XDOM5(P) $   Y = YDOM5(P)    $ P = 0 TO 20.
     6. X = 12.-P    $   Y = 0.0         $ P = 0 TO 12.
F.
        FUNCTION XDOM3(T)
        XDOM3 = 15.+ T
        IF( T .GE. 5.)            RETURN
           P1730 = AMAX1(AMIN1(T,1000.),-2.)
           XDOM3 = 20.-10.*COS(3.141592654*P1730*.1)
           RETURN
        END
        FUNCTION YDOM3(T)
        YDOM3 = 13.
        IF( T .GE. 5. )                      RETURN
           P1730 = AMAX1(AMIN1(T,1000.),-2.)
           YDOM3 = 23.-10.*SIN(3.141592654*P1730*.1)
           RETURN
        END
        FUNCTION XDOM5(T)
        XDOM5 = 34. - T
        X = 349 - 7
        IF( T .LE. 15. )                     RETURN
           P1755 = AMAX1(AMIN1(T,75.),-25.)
           XDOM5 = 19.-7.*SIN(3.141592654*(P1755-15.)*.1)
           RETURN
        END
        FUNCTION YDOM5(T)
        YDOM5 = 7.
        IF( T .LE. 15. )                     RETURN
           P1755 = AMAX1(AMIN1(T,75.),-25.)
           YDOM5 = 7.*COS(3.141592654*(P1755-15.)*.1)
           RETURN
        END
        FUNCTION TRUE(X,Y)
        TRUE = X*Y + Y -2.
        RETURN
        END
END.
```

Figure 2. Two domains with their specifications in ELLPACK.

```
*            ELLPACK 78 USER GUIDE EXAMPLE 3 - MAY 1979
*
EQUATION.      UXX$ + UYY = -1.
*
DOMAIN.        COUNTER-CLOCKWISE
     1. TO 3.  LINE  4.,4.  TO  1.,4.  TO 1.,.5 TO 4.,-.5
     4.  X = 4. + .1*P*(P-4.5)**2  $  Y =-.5 + P   $  P = 0. TO 4.5
```

DOMAIN OF SOLUTION




```
*            ELLPACK 78 USER GUIDE EXAMPLE 4 - MAY 1979
*
EQUATION.      GR(X,Y)*UXX$ + GR(X,Y)*UYY = 0.0
*
*                    HALF OF SQUARE WITH CIRCULAR HOLE
DOMAIN.
  1. X = P $ Y = .4 $ P = 0 TO .8  $ 2. X =.8  $ Y = .4-P  $ P = 0 TO .4
  3. X = .8-TAU $ Y = 0.0 $ TAU = 0.0 TO 0.2   $ 4.  X = .4+.2*COS(3.1415926*XI)
     Y = .2*SIN(3.1415926*XI) $  XI = 0.0 TO 1.0
  5. X = .2 - P12345  $ Y = 0.0  $  P12345 =  0.0000 TO 0.200000
  6. X = 0.  $ Y =  P $ P = 0.0 TO  0.4
*
FORTRAN.
      FUNCTION GR(X,Y)
      GR = -EXP(-(X+Y))
      RETURN
      END
      FUNCTION TRUE(X,Y)
      TRUE = EXP(X+Y)
      RETURN
      END
      FUNCTION D3EST(X,Y)
      D3EST=1.0
      RETURN
      END
 END.
```

DOMAIN OF SOLUTION

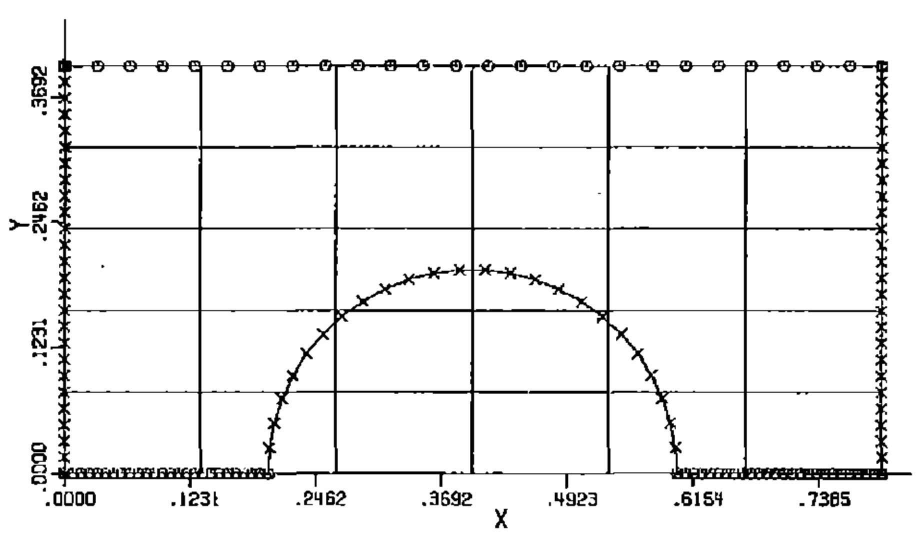


Figure 3. Two domains with their specifications in ELLPACK.

The protection is to use

SQRT(ABS (1.-T**2))

instead of the natural SQRT (1.-T**2) expression.

There are many ways that error conditions can be generated inadvertently; the main thing is to be aware of the possibilities and to use protective devices whenever a potential trouble is identified. The April, 1980 version of ELLPACK has no protection against the parameter being outside the designated range.

3. THE BOUNDARY SEGMENT.

The boundary conditions are specified in the BOUNDARY segment as a list of conditions associated with the list of boundary pieces. This list must be given in order. Thus

6. U= X+1.

specifies that on the sixth piece the given Dirichlet condition is to be satisfied. The form for giving Dirichlet, Neumann and Mixed boundary conditions are exactly the same as in ELLPACK 77 and the same keywords (DIRICHLET, NEUMANN, MIXED and HOMOGENEOUS) are used. Figure 4 shows example boundary condition specifications for the four domains given in Figures 2 and 3.

```
*
BOUNDARY. 1. U = TRUE(X,Y)
GRID.   UNIFORM X = 4, 0.0 TO 2.0 $  UNIFORM Y =  5,  0.0 TO 2.0
*
```

```
*
BOUND.    1. U = TRUE(X,Y)
          2. U = TRUE(X,Y)
          3. U = TRUE(X,Y)
          4. U = TRUE(X,Y)
          5. U = TRUE(X,Y)
          6. U = TRUE(X,Y)
*
GRID.    UNIFORM X =  9, 0.0 TO 34.
         UNIFORM Y = 6, 0.0 TO 24.
```

```
*
BOUND.
          1. U = EXP(1.+X)  $ 2. U = EXP(.8+Y)
          3. UY= EXP(X)     $ 4. U = TRUE(X,Y)
          5. UY= EXP(X)     $ 6. U = EXP(Y)
*
GRID.    UNIFORM Y =  6, 0.TO .4
         UNIFORM X =  7, 0. TO .8
*
```

```
BOUNDARY.
          1.  U = 4.-X   $  2.  U = 4.
          3.  U = 4.-X   $  4.  U = 0.0
*
GRID.  UNIFORM X = 12 ,   0.0 TO 5.5
       UNIFORM Y = 11 , -1. TO 4.0
```

Figure 4. The specifications of boundary conditions and grids for the domains of Figures 2 and 3.

4. THE GRID SEGMENT.

The GRID segment of ELLPACK 77 is changed in the way a uniform grid is specified. In ELLPACK 77 we have UNIFORM X=11 meaning that 11 grid lines are placed in the x-range of the domain. In ELLPACK 78 the x-range is not known (or is not well defined) so the range in which the uniform grid is placed must be specified as in:

UNIFORM X=11, 0.0 TO 5.0

The comma and the keyword TO separate the information about the number (11) of grid lines, the starting line (0.0) and the ending line (5.0). The same convention applies to the y coordinate and non-uniform grids are specified as in ELLPACK 77.

The ELLPACK system checks for the boundary going out of the grid defined and this is a fatal error.

5. Effects on OUTPUT.

There are new output commands for the internal information defining the boundary and domain. See Appendix A for a brief discussion of this information and for four examples. The new commands are

TABLE-DOMAIN

TABLE-BOUNDARY

The TABLE-DOMAIN command produces the GTYPE table which indicates the nature of all grid point (interior, exterior, boundary and next-to-boundary). Various information is encoded in the boundary and next-to-boundary entries.

The TABLE-BOUNDARY command produces the values of variables that define the boundary (see Appendix A for definitions). These include NBOUND (No. of boundary points), NGRID X and NGRID Y (No. of grid lines). There are seven arrays (XBOUND, YBOUND, PARAM, PIECE, BPTYPE, BGRID and BNEIGH) tabulated whose entries contain information pertinent to each boundary point.

The PLOT command applies in ELLPACK 78 for contour plots, but they might not be as accurate as desired. The reason is that it is a lengthy calculation to determine the intersection of contour lines with a curved boundary. Only an approximate calculation is made which may lead to contour lines stopping short of the boundary. Note that contours are plotted based on a 40 x 40 grid and their accuracy is further limited by this. See Figure 5 for an example contour plot.

6. THE HOLE SEGMENT

The HOLE segment allows for multiply connected domains; the form of the statement is exactly the same as DOMAIN except the keyword HOLE is used. Thus, for a square with a semi-circular hole, we have

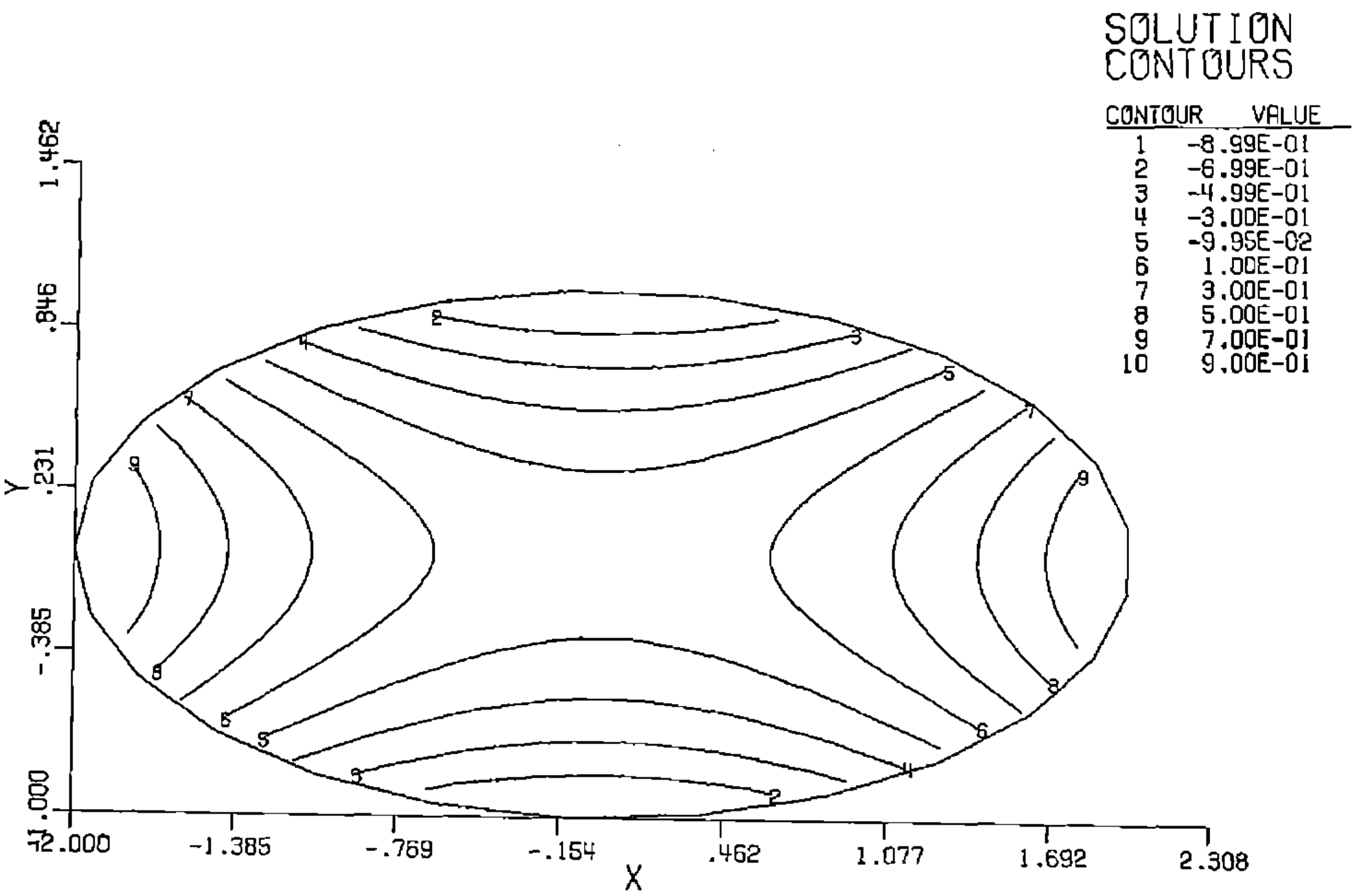


Figure 5. An example contour plot for a curved domain. Notice the lack of precision of the contour lines next to the boundary.

```
VERSION OF SEPTEMBER, 1978                    UPDATE OF MARCH 31, 1980
EQ.    UXX$ + UYY$ = 0.
*
*
*              ELLPACK 78 USERS GUIDE EXAMPLE  5 - JUNE 1980
*
DOMAIN.  1. TO 4.  LINE 0,.3 TO 0,1 TO 1,1 TO 1,.3 TO 0,.3
*
HOLE.
    5.  X=.5 +.2*SIN(T) $ Y = .55+.2*COS(T)  $  T = -1.57079633 TO 1.57079633
    6.  LINE .7,.55 TO .3,.55
BOUNDARY.
        1. U = 0.  $ 2. U = 0.  $  $ 3. U = 0.  $ 4. U = 0.
        5. U = 1.  $ 6. U = 1.
*
*
GRID.   UNIFORM X = 11, 0. TO 1.  $  UNIFORM Y = 8, .3 TO  1.
```

This facility does not have any effect on the ELLPACK modules, but it does require the user to be careful to make the grid fine enough. There should be at least two interior grid points separating the boundaries of the domain and the hole; even this is unlikely to give much accuracy in the solution.

The HOLE segment may appear several times to define more complex regions.

APPENDIX A: INTERNAL REPRESENTATION OF THE GEOMETRY INFORMATION

The basic geometrical situation in two dimensions is shown in Figure A-1. The relationship with a rectangular grid is shown in Figure A-2. Each piece is represented parametrically in the form $x = x(p)$, $y = y(p)$ for p in [a,b].

3.3 Interface 2: Domain Representation.

The objectives of the domain processing module are:

a. locate the domain boundary with respect to the grid.

b. relate all grid points to the domain (e.g. interior or exterior)

c. provide this information in various forms so that later processing of the domain requires a minimum of geometric analysis.

The information generated is thus of two types. One is associated with the boundary intersections with the grid and "follows the boundary around the domain". The other is various information encoded in GTYPE associated with the two dimensional array of grid points. The specific information in GTYPE is given below

```
GTYPE(IX,IY)  FOR IX = 1 TO NGRIDX, IY = 1 TO NGRIDY
 =   0       GRID POINT OUTSIDE DOMAIN, AWAY FROM BOUNDARY.
 = INSIDE    GRID POINT  INSIDE DOMAIN, AWAY FROM BOUNDARY.
             INSIDE = CONSTANT INTEGER, E.G. 999
 = INTEGER   LESS THAN INSIDE, GRID PT = BOUNDARY PT OF INDEX GTYPE
 = INTEGER   GREATER THAN INSIDE+1 IN ABSOLUTE VALUE
             GRID POINT IS NEXT TO THE BOUNDARY. RELATION TO THE
             BOUNDARY IS ENCODED BY
                     GTYPE = INDEX + IPACKB*J
             WHERE
                INDEX  = SMALLEST INDEX OF NEIGHBORING BOUNDARY PT
                IPACKB = CONSTANT FOR PACKING = INSIDE + 1
                J      = FOUR BITS FOR DIRECTION TO BOUNDARY PTS.
                           =1 FOR NOON       = 2 FOR 3 OCLOCK
                           =4 FOR 6 OCLOCK, = 8 FOR 9 OCLOCK
                           THUS J = 9 = 1001(BINARY) SHOWS BOUNDARY
                           POINTS ABOVE AND TO THE LEFT.
 = POSITIVE FOR GRID POINT IN THE DOMAIN OR ON BOUNDARY
 = NEGATIVE OR 0 FOR POINT OUTSIDE THE DOMAIN
```

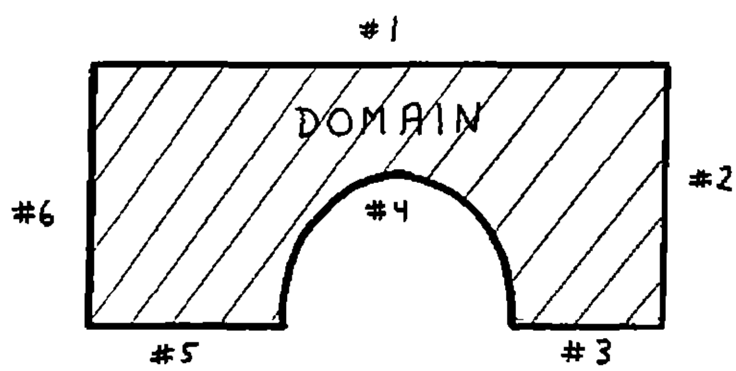


Figure A-1. A typical two-dimensional rectangular domain. The pieces of the boundary are numbered clockwise and described parametrically.

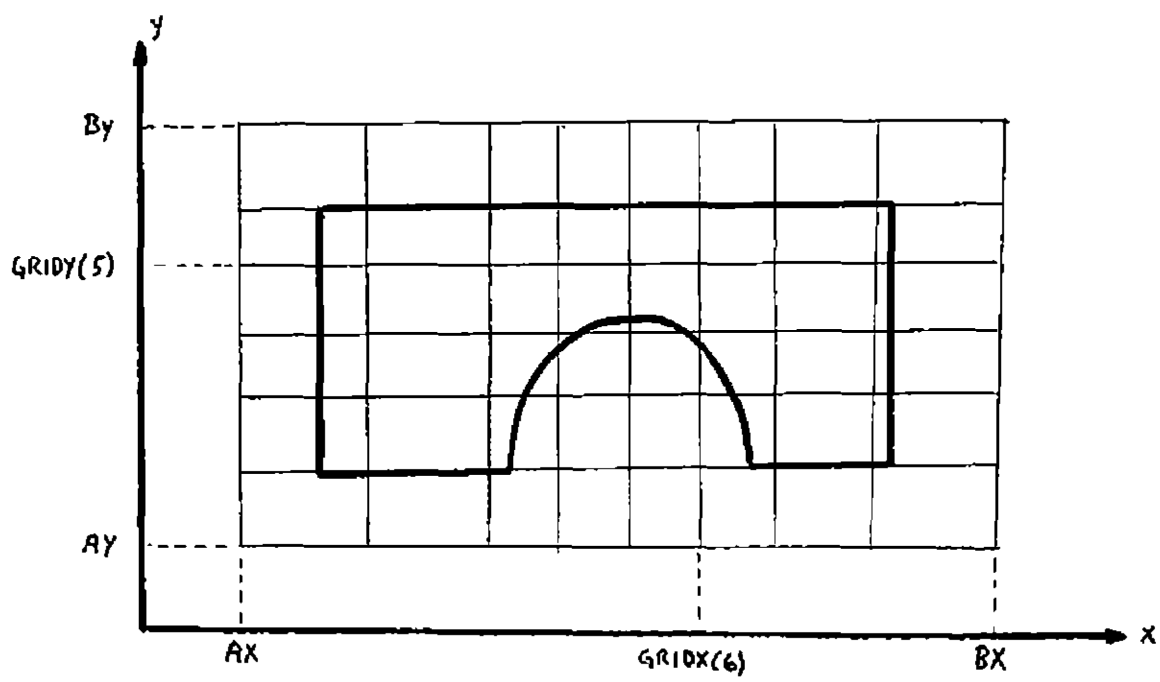


Figure A-2. The domain of Figure A-1 with a rectangular grid overlay. Here we have NGRIDX = 9 and NGRIDY = 7. In practice one would expect the horizontal and vertical sides to be grid lines.

The information about the intersection of the boundary with the grid lines is contained in a set of 7 linear arrays of length NBNDPT. These arrays all have problem dependent dimensions MAXBND and are used and defined as shown below

```
INTEGER PIECE(MAXBND),BPTYPE(MAXBND),BNEIGH(MAXBND),BGRID(MAXBND)
REAL            XBOUND(MAXBND),YBOUND(MAXBND),BPARAM(MAXBND)

    XBOUND(I),YBOUND(I) = COORDINATES OF I-TH BOUNDARY POINT
    BPARAM(I) = PARAMETER VALUE P OF I-TH BOUNDARY POINT
    PIECE (I) = INDEX OF BOUNDARY PIECE TO WHICH POINT BELONGS
                SMALLEST NUMBER FOR CORNER POINTS
    BPTYPE(I) = THE TYPE OF THE BOUNDARY POINT
              = HORZ   IF POINT IS ON A Y GRID LINE
              = VERT   IF POINT IS ON A X GRID LINE
              = BOTH   IF POINT IS ALSO A GRID POINT
              = CORN   IF POINT IS A CORNER PT ON A GRID LINE
              = INTE   IF POINT IS NOT ON A GRID LINE
                       HAPPENS ONLY FOR CORNERS NOT ON THE GRID
                THE CONSTANTS HORZ, VERT, BOTH, CORN   AND INTER
                 ARE PLACED BY ELLPACK CONTROL IN COMMON / INTEGS/
    BNEIGH(I) = IX + 1000*IY  WHERE
                 IX,IY = INDEXES OF THE X-GRID AND Y-GRID LINES
                        OF THE FIRST CLOCKWISE INTERIOR GRID
                        POINT ADJACENT TO THIS BOUNDARY POINT.
              = I   IF THE POINT IS ALSO A GRID POINT
              = 0   IF THE POINT IS INTERIOR( NOT ON THE GRID )
    BGRID (I) = IX + 1000*IY  WHERE THE I-TH BOUNDARY POINT
                IS IN THE GRID SQUARE IX,IY. THAT IS
                   GRIDX(IX) .LE.  XBOUND(I) .LT. GRIDX(IX+1)
                   GRIDY(IY) .LE.  YBOUND(I) .LT. GRIDY(IY+1)
               (NOTE. IX,IY ARE LIMITED TO NGRIDX, NGRIDY)
```

Figures A-3 through A-6 give the information for representing the four domains of Figures 2 and 3.

THE 10 BOUNDARY POINTS IN THE 4 BY 5 GRID

I	X	Y	PARAM	PIECE	TYPE	GRID	NEIGH
1	0	1.00000E+00	0	1	CORN	3001	0
2	1.33975E-01	1.50000E+00	1.66667E-01	1	HORZ	4001	4002
3	6.66667E-01	1.94281E+00	3.91827E-01	1	VERT	4002	4002
4	1.33333E+00	1.94281E+00	6.08173E-01	1	VERT	4003	4003
5	1.86603E+00	1.50000E+00	8.33333E-01	1	HORZ	4003	4003
6	2.00000E+00	1.00000E+00	1.00000E+00	1	BOTH	3004	3003
7	1.86603E+00	5.00000E-01	1.16667E+00	1	HORZ	2003	2003
8	1.33333E+00	5.71910E-02	1.39183E+00	1	VERT	1003	2003
9	6.66667E-01	5.71910E-02	1.60817E+00	1	VERT	1002	2002
10	1.33975E-01	5.00000E-01	1.83333E+00	1	HORZ	2001	2002

THE GRID POINT TYPES WITH (XGRID(1),YGRID(1)) AT LOWER LEFT

```
5 *       0  -4003  -4004      0
4 *   -6001   9002   3004 -12005
3 *       1   8001   2006      6
2 *   -3001  12009   6007  -9006
1 *       0  -1009  -1008      0
  *
  ******* 1      2      3      4
```

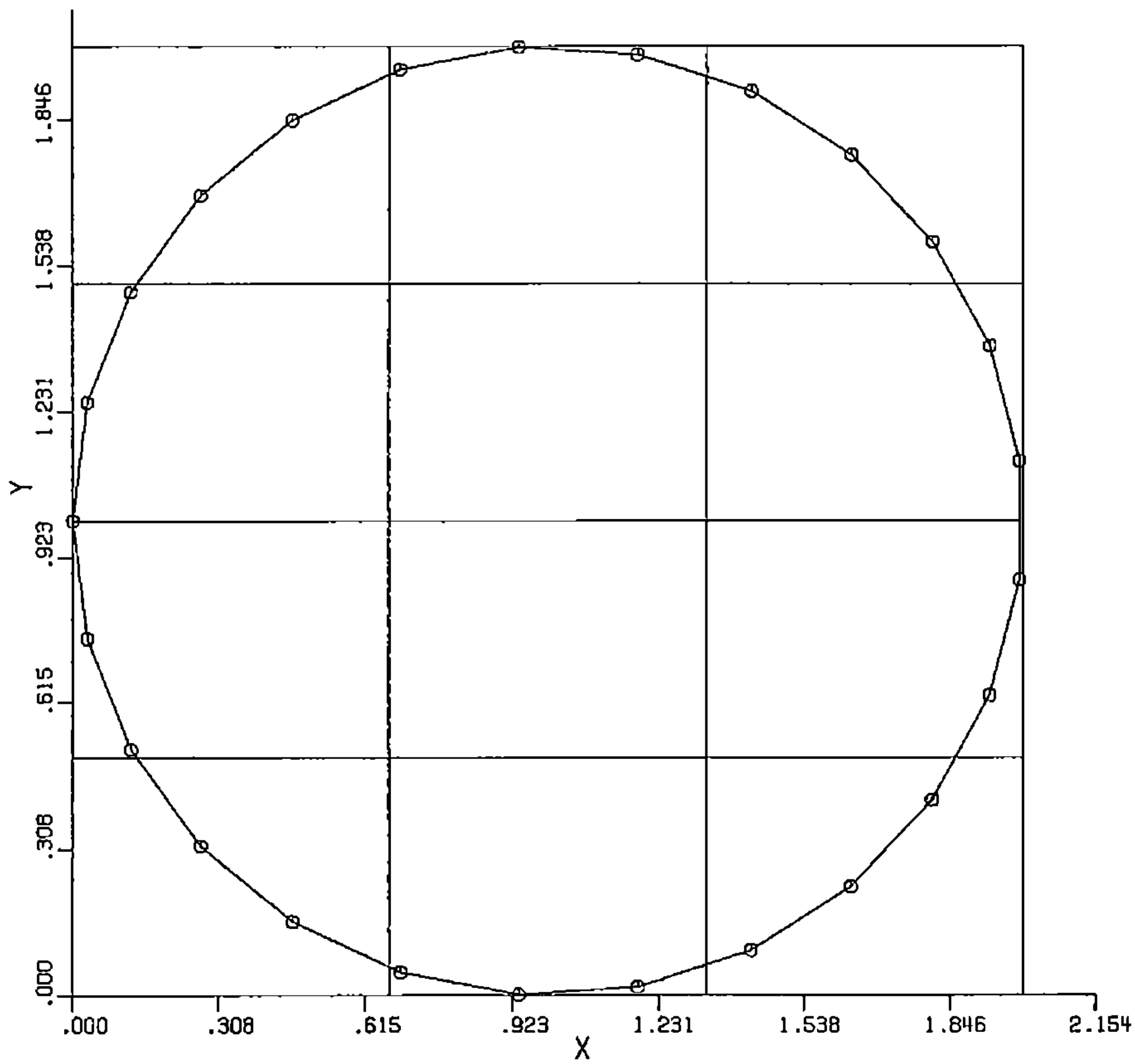


Figure A-3. The boundary/grid information for Example 1.

THE 28 BOUNDARY POINTS IN THE 9 BY 6 GRID

I	X	Y	PARAM	PIECE	TYPE	GRID	NEIGH
1	0	0	0	1	CORN	1001	0
2	0	4.80000E+00	2.08696E-01	1	BOTH	2001	2002
3	0	9.60000E+00	4.17391E-01	1	BOTH	3001	3002
4	0	1.44000E+01	6.26087E-01	1	BOTH	4001	4002
5	0	1.92000E+01	8.34783E-01	1	BOTH	5001	5002
6	0	2.30000E+01	1.00000E+00	1	CORN	5001	0
7	4.25000E+00	2.30000E+01	4.25000E-01	2	VERT	5002	5002
8	8.50000E+00	2.30000E+01	8.50000E-01	2	VERT	5003	5003
9	1.00000E+01	2.30000E+01	1.00000E+00	2	INTE	5003	0
10	1.07501E+01	1.92000E+01	1.24076E+00	3	HORZ	5003	5003
11	1.27500E+01	1.61125E+01	2.41840E+00	3	VERT	4004	4004
12	1.48971E+01	1.44000E+01	3.29537E+00	3	HORZ	4004	4004
13	1.70000E+01	1.34606E+01	4.03013E+00	3	VERT	3005	3005
14	2.12500E+01	1.30000E+01	6.25000E+00	3	VERT	3006	3006
15	2.55000E+01	1.30000E+01	1.05000E+01	3	VERT	3007	3007
16	2.97500E+01	1.30000E+01	1.47500E+01	3	VERT	3008	3008
17	3.40000E+01	1.30000E+01	1.90000E+01	3	CORN	3009	0
18	3.40000E+01	9.60000E+00	5.66667E-01	4	BOTH	3009	3008
19	3.40000E+01	7.00000E+00	1.00000E+00	4	CORN	2009	0
20	2.97500E+01	7.00000E+00	4.25000E+00	5	VERT	2008	3008
21	2.55000E+01	7.00000E+00	8.50000E+00	5	VERT	2007	3007
22	2.12500E+01	7.00000E+00	1.27500E+01	5	VERT	2006	3006
23	1.70000E+01	6.70820E+00	1.59223E+01	5	VERT	2005	3005
24	1.39049E+01	4.80000E+00	1.75949E+01	5	HORZ	2004	2004
25	1.27500E+01	3.15238E+00	1.85130E+01	5	VERT	1004	2004
26	1.20000E+01	0	2.00000E+01	5	CORN	1003	0
27	8.50000E+00	0	3.50000E+00	6	BOTH	1003	2003
28	4.25000E+00	0	7.75000E+00	6	BOTH	1002	2002

THE GRID POINT TYPES WITH (XGRID(1),YGRID(1)) AT LOWER LEFT

	1	2	3	4	5	6	7	8	9
6 *	-4006	-4007	-4008	0	0	0	0	0	0
5 *	5	9005	3008	-12010	0	0	0	0	0
4 *	4	8004	999	3011	-12012	-4014	-4015	-4016	-4017
3 *	3	8003	999	999	5013	5014	5015	7016	18
2 *	2	12002	4027	6024	-9023	-1022	-1021	-1020	-1019
1 *	1	28	27	-9025	0	0	0	0	0

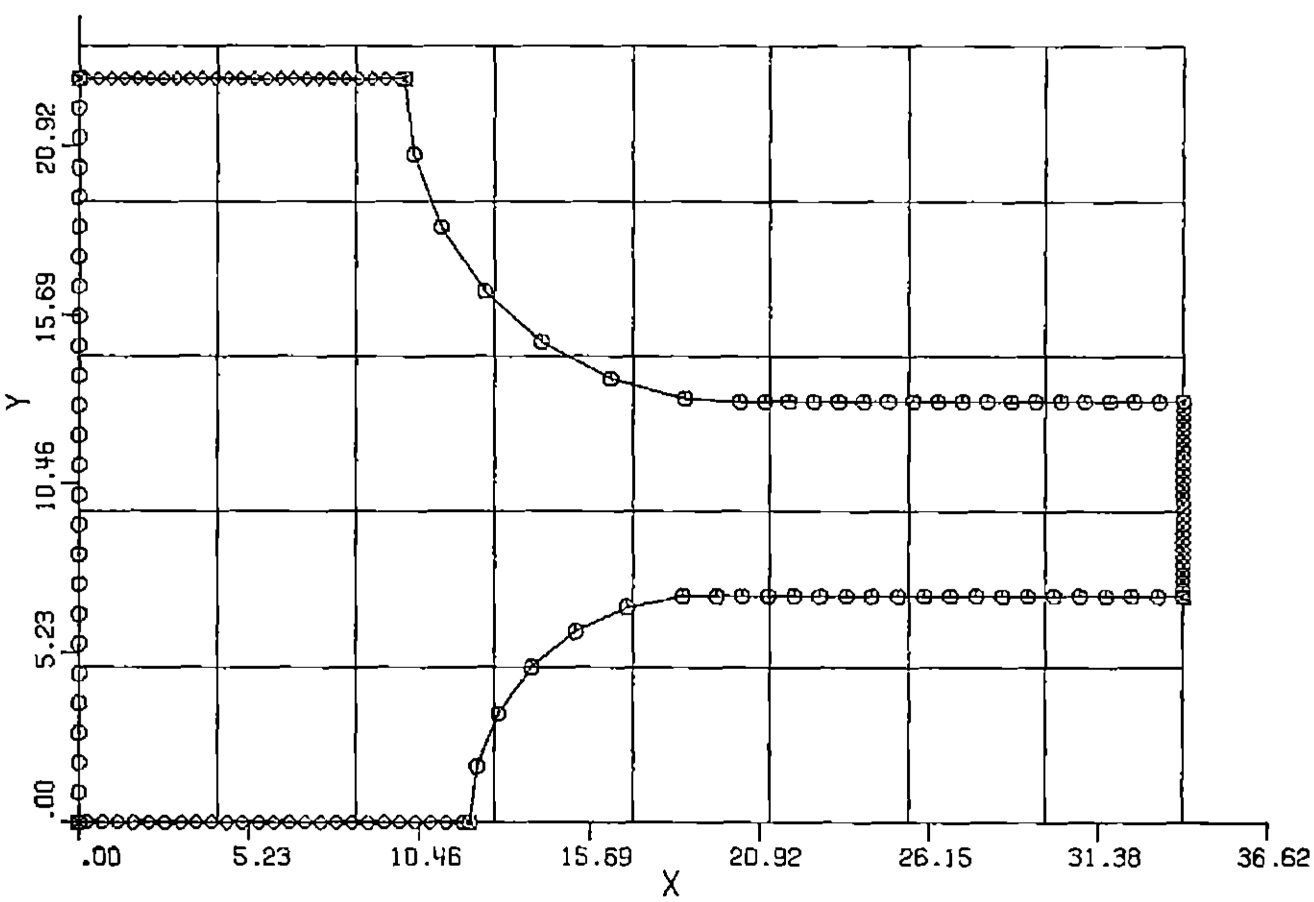


Figure A-4. The boundary/grid information for Example 2.

THE 31 BOUNDARY POINTS IN THE 12 BY 11 GRID

I	X	Y	PARAM	PIECE	TYPE	GRID	NEIGH
1	4.00000E+00	4.00000E+00	0	1	CORN	11009	0
2	3.50000E+00	4.00000E+00	1.66667E-01	1	BOTH	11008	10008
3	3.00000E+00	4.00000E+00	3.33333E-01	1	BOTH	11007	10007
4	2.50000E+00	4.00000E+00	5.00000E-01	1	BOTH	11006	10006
5	2.00000E+00	4.00000E+00	6.66667E-01	1	BOTH	11005	10005
6	1.50000E+00	4.00000E+00	8.33333E-01	1	BOTH	11004	10004
7	1.00000E+00	4.00000E+00	1.00000E+00	1	CORN	11003	0
8	1.00000E+00	3.50000E+00	1.42857E-01	2	BOTH	10003	10004
9	1.00000E+00	3.00000E+00	2.85714E-01	2	BOTH	9003	9004
10	1.00000E+00	2.50000E+00	4.28571E-01	2	BOTH	8003	8004
11	1.00000E+00	2.00000E+00	5.71429E-01	2	BOTH	7003	7004
12	1.00000E+00	1.50000E+00	7.14286E-01	2	BOTH	6003	6004
13	1.00000E+00	1.00000E+00	8.57143E-01	2	BOTH	5003	5004
14	1.00000E+00	5.00000E-01	1.00000E+00	2	CORN	4003	0
15	1.50000E+00	3.33333E-01	1.66667E-01	3	VERT	3004	4004
16	2.00000E+00	1.66667E-01	3.33333E-01	3	VERT	3005	4005
17	2.50000E+00	0	5.00000E-01	3	BOTH	3006	4006
18	3.00000E+00	-1.66667E-01	6.66667E-01	3	VERT	2007	3007
19	3.50000E+00	-3.33333E-01	8.33333E-01	3	VERT	2008	3008
20	4.00000E+00	-5.00000E-01	1.00000E+00	3	CORN	2009	0
21	4.50000E+00	-2.19116E-01	2.80884E-01	4	VERT	2010	3010
22	4.80000E+00	0	5.00000E-01	4	HORZ	3010	3010
23	5.00000E+00	1.88262E-01	6.88262E-01	4	VERT	3011	4011
24	5.22500E+00	5.00000E-01	1.00000E+00	4	HORZ	4011	4011
25	5.35000E+00	1.00000E+00	1.50000E+00	4	HORZ	5011	5011
26	5.25000E+00	1.50000E+00	2.00000E+00	4	HORZ	6011	6011
27	5.00000E+00	2.00000E+00	2.50000E+00	4	BOTH	7011	7010
28	4.67500E+00	2.50000E+00	3.00000E+00	4	HORZ	8010	8010
29	4.50000E+00	2.76192E+00	3.26192E+00	4	VERT	8010	8010
30	4.35000E+00	3.00000E+00	3.50000E+00	4	HORZ	9009	9009
31	4.10000E+00	3.50000E+00	4.00000E+00	4	HORZ	10009	10009

THE GRID POINT TYPES WITH (XGRID(1),YGRID(1)) AT LOWER LEFT

11 *	0	-2007	7	6	5	4	3	2	1	-8001	0	0
10 *	0	-2008	8	9006	1005	1004	1003	1002	3001	-8031	0	0
9 *	0	-2009	9	8009	999	999	999	999	2030	-12029	0	0
8 *	0	-2010	10	8010	999	999	999	999	999	3028	-12027	0
7 *	0	-2011	11	8011	999	999	999	999	999	2027	27	-8027
6 *	0	-2012	12	8012	999	999	999	999	999	999	3026	-8026
5 *	0	-2013	13	8013	999	999	999	999	999	999	2025	-8025
4 *	0	-2014	14	12014	4016	4017	999	999	999	999	6023	-8024
3 *	0	0	-1014	-1015	-3016	17	12017	4019	4020	6021	-9022	0
2 *	0	0	0	0	0	-1017	-1018	-3019	20	-9020	0	0
1 *	0	0	0	0	0	0	0	0	-1020	0	0	0
*												
*******	1	2	3	4	5	6	7	8	9	10	11	12

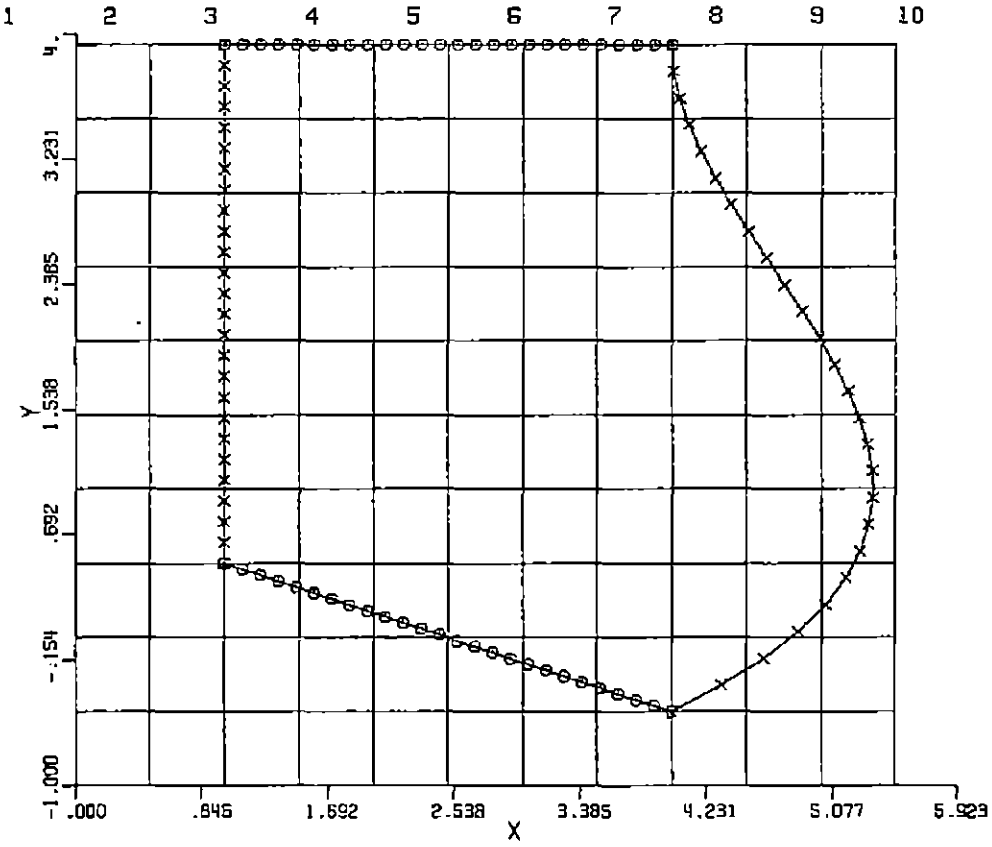


Figure A-5. The boundary/grid information for Example 3.

THE 47 BOUNDARY POINTS IN THE 11 BY 8 GRID

I	X	Y	PARAM	PIECE	TYPE	GRID	NEIGH
1	0	3.00000E-01	0	1	CORN	1001	0
2	0	4.00000E-01	1.42857E-01	1	BOTH	2001	2002
3	0	5.00000E-01	2.85714E-01	1	BOTH	3001	3002
4	0	6.00000E-01	4.28571E-01	1	BOTH	4001	4002
5	0	7.00000E-01	5.71429E-01	1	BOTH	5001	5002
6	0	8.00000E-01	7.14286E-01	1	BOTH	6001	6002
7	0	9.00000E-01	8.57143E-01	1	BOTH	7001	7002
8	0	1.00000E+00	1.00000E+00	1	CORN	8001	0
9	1.00000E-01	1.00000E+00	1.00000E-01	2	BOTH	8002	7002
10	2.00000E-01	1.00000E+00	2.00000E-01	2	BOTH	8003	7003
11	3.00000E-01	1.00000E+00	3.00000E-01	2	BOTH	8004	7004
12	4.00000E-01	1.00000E+00	4.00000E-01	2	BOTH	8005	7005
13	5.00000E-01	1.00000E+00	5.00000E-01	2	BOTH	8006	7006
14	6.00000E-01	1.00000E+00	6.00000E-01	2	BOTH	8007	7007
15	7.00000E-01	1.00000E+00	7.00000E-01	2	BOTH	8008	7008
16	8.00000E-01	1.00000E+00	8.00000E-01	2	BOTH	8009	7009
17	9.00000E-01	1.00000E+00	9.00000E-01	2	BOTH	8010	7010
18	1.00000E+00	1.00000E+00	1.00000E+00	2	CORN	8011	0
19	1.00000E+00	9.00000E-01	1.42857E-01	3	BOTH	7011	7010
20	1.00000E+00	8.00000E-01	2.85714E-01	3	BOTH	6011	6010
21	1.00000E+00	7.00000E-01	4.28571E-01	3	BOTH	5011	5010
22	1.00000E+00	6.00000E-01	5.71429E-01	3	BOTH	4011	4010
23	1.00000E+00	5.00000E-01	7.14286E-01	3	BOTH	3011	3010
24	1.00000E+00	4.00000E-01	8.57143E-01	3	BOTH	2011	2010
25	1.00000E+00	3.00000E-01	1.00000E+00	3	CORN	1011	0
26	9.00000E-01	3.00000E-01	1.00000E-01	4	BOTH	1010	2010
27	8.00000E-01	3.00000E-01	2.00000E-01	4	BOTH	1009	2009
28	7.00000E-01	3.00000E-01	3.00000E-01	4	BOTH	1008	2008
29	6.00000E-01	3.00000E-01	4.00000E-01	4	BOTH	1007	2007
30	5.00000E-01	3.00000E-01	5.00000E-01	4	BOTH	1006	2006
31	4.00000E-01	3.00000E-01	6.00000E-01	4	BOTH	1005	2005
32	3.00000E-01	3.00000E-01	7.00000E-01	4	BOTH	1004	2004
33	2.00000E-01	3.00000E-01	8.00000E-01	4	BOTH	1003	2003
34	1.00000E-01	3.00000E-01	9.00000E-01	4	BOTH	1002	2002
35	0	3.00000E-01	1.00000E+00	4	JUMP	1001	0
36	3.00000E-01	5.50000E-01	-1.57080E+00	5	CORN	3004	0
37	3.06351E-01	6.00000E-01	-1.31812E+00	5	HORZ	4004	4005
38	3.67712E-01	7.00000E-01	-7.22734E-01	5	HORZ	5004	5005
39	4.00000E-01	7.23205E-01	-5.23599E-01	5	VERT	5005	5005
40	5.00000E-01	7.50000E-01	-5.61976E-12	5	VERT	5006	5006
41	6.00000E-01	7.23205E-01	5.23599E-01	5	VERT	5007	5007
42	6.32288E-01	7.00000E-01	7.22734E-01	5	HORZ	5007	5007
43	6.93649E-01	6.00000E-01	1.31812E+00	5	HORZ	4007	4007
44	7.00000E-01	5.50000E-01	1.57080E+00	5	CORN	3008	0
45	6.00000E-01	5.50000E-01	2.50000E-01	6	VERT	3007	4007
46	5.00000E-01	5.50000E-01	5.00000E-01	6	VERT	3006	4006
47	4.00000E-01	5.50000E-01	7.50000E-01	6	VERT	3005	4005

ELLPACK 78 OUTPUT

TABLE OF THE POINT TYPES ON 11 X 8 GRID

THE POINT XGRID(1), YGRID(1) IS AT LOWER LEFT.

	1	2	3	4	5	6	7	8	9	10	11
8	8	9	10	11	12	13	14	15	16	17	18
7	7	9007	1010	1011	1012	1013	1014	1015	1016	3017	19
6	6	8006	999	999	4039	4040	4041	999	999	2020	20
5	5	8005	999	2038	-9038	-1040	-3041	8042	999	2021	21
4	4	8004	999	6036	-12037	-4046	-6043	12043	999	2022	22
3	3	8003	999	1036	1047	1046	1045	1044	999	2023	23
2	2	12002	4033	4032	4031	4030	4029	4028	4027	6024	24
1	1	34	33	32	31	30	29	28	27	26	25

Figure A-6. The boundary/grid information for Example 5.

APPENDIX B: ELLPACK 78 DISCRETIZATION MODULES.

MODULE NAME: 5-POINT STAR

AUTHOR/DATE: Ron Boisvert, June 1977 (revised Nov. 1978)

INITIAL/FINAL INTERFACES: EQUATION FORMATION - EQUATION INDEXING

MODULE FUNCTIONS: Discretizes a general linear elliptic PDE with boundary conditions of the form $pu_x + qu_y + r = g$.

RESTRICTIONS ON USE: Two dimensions, no U_{xy} term, uniform grid, non-self-adjoint form. Caution: this is an experimental version of the 5-POINT STAR code; no claims are made about its correctness.

METHOD DESCRIPTION: The usual 5-POINT finite difference discretization is used in the interior of the domain. Adjacent to the boundary a lower order non-symmetric divided difference approximation is used. The derivatives in boundary conditions are approximated by three point one-sided differences, with values at non-grid

PARAMETERS: None

KEYWORDS THAT AFFECT MODULE: CONSTANT COEFFICIENTS

STORAGE AND TIMING ESTIMATES: Approximately NGRIDX * NGRIDY equations are generated with up to 7 unknowns per equation. A workspace of 25 + NGRIDX + NGRIDY + number of boundary-grid intersections is used.

MODULE NAME: 2DEPEP

AUTHOR/DATE: G. Sewell, 10/16/78

INITIAL/FINAL INTERFACE: EQUATION FORMATION-OUTPUT

RESTRICTIONS: Two-dimensions, self-adjoint

MODULE DESCRIPTION: Galerkin's method with 6-node quadratic triangular elements, user-controlled grading of the triangular mesh, and the frontal method to organize out-of-core storage of the matrix when necessary. This is a small, specialized subset of the program TWODEPEP, a commercial product of IMSL, Inc. For further details see G. Sewell, A finite element program with automatic user controlled mesh grading B-2, in Advances in Computer Methods for Partial Differential Equations III (R. Vishnevetsky, ed.) Rutgers Univ., New Brunswick, NJ.

PARAMETERS: NTRI- number of triangles desired in final triangulation
MEM- workspace storage = 71*NTRI if external storage to be used
46*NTRI + 15*NTRI**1.5 otherwise

The grid is used to define an initial triangulation. This triangulation will have about 4 triangles for each grid square which intersects the region, so NTRI must be larger than this number. The closure of the intersection of any grid square with the region must be convex or nearly so. Thus it is necessary, in general, that any region corner with exterior angle less than 180^o be cut by a grid line which divides the exterior angle into two parts. (In the case of a 90^o exterior angle with edges parallel to the axes, however, it is sufficient to put a grid point at that corner.)

FORTRAN: Function D3EST(X,Y) must be user-supplied.
The program grades the initial triangulation so that the final triangulation is most dense where D3EST is largest. In particular, it attempts to distribute D3EST(X_j,Y_j)*H_j**3 uniformly, where (X_j,Y_j) is the center of triangle j and H_j is its diameter. If D3EST is an estimate of the function

$$\max_{i+j=3} \left| D_x^i D_y^j u \right|$$

it is possible to obtain optimal order convergence to the solution of some singular problems.